Measure Theory with Ergodic Horizons
Lecture 21

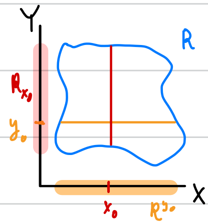
Fubini–Tonelli theorem.

Def. Let X, Y, Z be sets and $x_0 \in X$, $y_0 \in Y$.

(a) For a set $R \subseteq X \times Y$, call the sets

$$R_{x_0} := \{y \in Y : (x_0, y) \in R\} \subseteq Y$$

$$R^{y_0} := \{x \in X : (x, y_0) \in R\} \subseteq X$$



the vertical fiber of R at x_0 and the horizontal fiber of R at y_0, respectively.

(b) For a function $f : X \times Y \to Z$, call the functions

$$f_{x_0} : Y \to Z \qquad \text{and} \qquad f^{y_0} : X \to Z$$
$$y \mapsto f(x_0, y) \qquad\qquad\qquad x \mapsto f(x, y_0)$$

the vertical fiber of f at x_0 and the horizontal fiber of f at y_0, respectively.

Theorem (Fubini–Tonelli). Let $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$ be σ-finite measure spaces.
Let $f : X \times Y \to \overline{\mathbb{R}} := [-\infty, \infty]$ be a $\mu \times \nu$-measurable function. Then:

(a) $f_x : Y \to \overline{\mathbb{R}}$ and $f^y : X \to \overline{\mathbb{R}}$ are ν and μ-measurable for μ-a.e. $x \in X$ and ν-a.e. $y \in Y$

(b) Tonelli. If $f \geq 0$, then:

(i) $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are μ and ν-measurable.

(ii) $$\int_X \int_Y f_x(y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \int_X f^y(x)\, d\mu(x)\, d\nu(y).$$

(c) Fubini. If f is $\mu\times\nu$-integrable then:

(i) $x \mapsto \int_Y f_x\, d\nu$ and $y \mapsto \int_X f^y\, d\mu$ are μ and ν-measurable and integrable.

(ii) $$\int_X \int_Y f_x(y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \int_X f^y(x)\, d\mu(x)\, d\nu(y).$$

Remark. All conditions in this theorem are necessary; examples will be given in HW.

Remark. Usually, one first applies Tonelli to $|f|$ to show that f is integrable, and only afterwards applies Fubini to f.

Example. For sets N and M (e.g. $N = M = \mathbb{N}$) and counting measures μ and ν on M and N, respectively, we already know the Fubini-Tonelli from basic real analysis:

(a) Tonelli. If $(a_{nm})_{(n,m)\in N\times M}$ is a matrix of nonnegative reals, then
$$\sum_{n\in N}\sum_{m\in M} a_{nm} = \sum_{(n,m)\in N\times M} a_{nm} = \sum_{m\in M}\sum_{n\in N} a_{nm}.$$

(b) Fubini. If $(a_{nm})_{(n,m)\in N\times M}$ is an absolutely summable matrix of reals, i.e. $\sum_{(n,m)\in N\times M} |a_{nm}| < \infty$, then
$$\sum_{n\in N}\sum_{m\in M} a_{nm} = \sum_{(n,m)\in N\times M} a_{nm} = \sum_{m\in M}\sum_{n\in N} a_{nm}.$$

We will prove the Fubini-Tonelli theorem for $\mathcal{I}\otimes\mathcal{J}$-measurable functions, replacing the μ- and ν-measurability in the conclusion with $\mathcal{I}$- and $\mathcal{J}$-measurability. Deducing the theorem for $\mu\times\nu$-measurable functions is left as a HW exercise.

Prop. Let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be measurable spaces.

(a) For any $\mathcal{I}\otimes\mathcal{J}$-measurable set $R\subseteq X\times Y$, i.e. $R\in\mathcal{I}\otimes\mathcal{J}$, we have that all fibers R_x and R^y are $\mathcal{J}$ and $\mathcal{I}$-measurable, respectively.

(b) For any $\mathcal{I}\otimes\mathcal{J}$-measurable function $f: X\times Y\to Z$, where $(Z,\mathcal{K})$ is a measurable space, all fibers f_x and f^y are $\mathcal{J}$ and $\mathcal{I}$-measurable, respectively.

Proof. (a) Let $\mathcal{C} := \{R\in\mathcal{I}\otimes\mathcal{J} : \forall x\in X\ \forall y\in Y,\ f_x \text{ and } f^y \text{ are measurable}\}$. By definition, $\mathcal{C}$ contains all rectangles $U\times V$, where $U\in\mathcal{I}$ and $V\in\mathcal{J}$. Moreover, $\mathcal{C}$ is closed under complement and ctbl unions just because these operations commute with taking fibers: $(R^c)_x = R^c_x$ and $\left(\bigcup_n R_n\right)_x = \bigcup_n (R_n)_x$, same for horizontal fibers. Thus, $\mathcal{C}$ is a σ-algebra containing $\mathcal{I}\times\mathcal{J}$, hence $\mathcal{C}=\mathcal{I}\otimes\mathcal{J}$.

(b) Recall that $f: X\times Y\to Z$ is $\mathcal{I}\otimes\mathcal{J}$-measurable if $f^{-1}(W)\in\mathcal{I}\otimes\mathcal{J}$ for each $W\in\mathcal{K}$. But preimages commute with taking fibers, i.e. $f_x^{-1}(W) = (f^{-1}(W))_x$ and $(f^y)^{-1}(W) = (f^{-1}(W))^y$, so the claim follows from (a) since $f^{-1}(W)\in\mathcal{I}\otimes\mathcal{J}$. $\square$

Fubini-Tonelli for sets. Let $(X,\mathcal{I},\mu)$ and $(Y,\mathcal{J},\nu)$ be σ-finite measure spaces. Let $R\in\mathcal{I}\otimes\mathcal{J}$. Then:

(b) $x \mapsto \nu(R_x)$ and $y \mapsto \mu(R^y)$ are $\mathcal{I}$ and $\mathcal{J}$-measurable, respectively.

(c) $\int_X \nu(R_x)\, d\mu(x) = \mu \times \nu(R) = \int_Y \mu(R^y)\, d\nu(y)$.

Proof. Again let $\mathcal{C} := \{R \in \mathcal{I} \otimes \mathcal{J} : (b) \text{ and } (c) \text{ hold for } R\}$. Note that $\mathcal{C} \supseteq \mathcal{I} \times \mathcal{J}$, i.e. $\mathcal{C}$ contains each rectangle $U \times V$, because the functions $x \mapsto \nu((U \times V)_x) = \begin{cases} \nu(V) & \text{if } x \in U \\ 0 & \text{o.w.} \end{cases}$ $= \nu(V) \cdot \mathbb{1}_U(x)$ and $y \mapsto \mu((U \times V)^y) = \mu(U) \cdot \mathbb{1}_V(y)$ are just constant multiples of indicator functions of measurable sets, and hence are measurable. Furthermore, $\int_X \nu((U \times V)_x)\, d\mu(x) = \int_X \nu(V) \cdot \mathbb{1}_U(x)\, d\mu(x) = \nu(V) \cdot \mu(U) = \mu \times \nu(U \times V)$, similarly, for the horizontal fibers.

It is enough to prove assuming μ and ν are finite by the usual argument of writing $X = \bigsqcup_{n \in \mathbb{N}} X_n$ and $Y = \bigsqcup_{m \in \mathbb{N}} Y_m$ for some measurable finite measure sets X_n, Y_m, so $X \times Y = \bigsqcup_{n,m \in \mathbb{N}} X_n \times Y_m$ and $R = \bigsqcup_{n,m \in \mathbb{N}} (R \cap X_n \times Y_m)$.

Using the finiteness of μ and ν, it is not hard to verify that $\mathcal{C}$ is closed under complements. $\mathcal{C}$ is also closed under finite disjoint unions by the linearity of the integral: if $R = \bigsqcup_{i \leq n} R_i$ then $\mathbb{1}_R = \sum_{i \leq n} \mathbb{1}_{R_i}$, and the linearity of the integral applies. This implies that $\mathcal{C}$ contains the algebra $\mathcal{A}$ generated by the rectangles because finite unions of rectangles are finite disjoint unions of (different) rectangles. But this doesn't imply closedness under finite nondisjoint unions because $R = R_1 \cup R_2$ then $\mathbb{1}_R = \mathbb{1}_{R_1} + \mathbb{1}_{R_2} - \mathbb{1}_{R_1 \cap R_2}$.

Let's ask ourselves if $R = \bigcup_{n\in\mathbb{N}} R_n$, then is $x \mapsto \nu(R_x)$ a limit of $x \mapsto \nu((R_n)_x)$? Indeed the answer is when the union is monotone. This shows that $\mathcal{C}$ is closed under

- ctbl increasing unions: $R = \bigcup\!\!\!\!\uparrow_n R_n$, then by the increasing monotonicity of measure, we have that $(x \mapsto \nu((R_n)_x)) \nearrow (x \mapsto \nu(R_x))$ and by MCT, we have
$$\int \nu(R_x)\, d\mu(x) = \lim_{n\to\infty} \int \nu((R_n)_x)\, d\mu(x) = \lim_{n\to\infty} \mu\times\nu(R_n) = \mu\times\nu(R),$$
and same for the horizontal fiber.

- decreasing intersections: $R = \bigcap\!\!\!\!\downarrow_n R_n$, then we use the finiteness of the measures μ, ν, and $\mu\times\nu$ to replace in the above argument increasing monotonicity with decreasing monotonicity and MCT with DCT.

To summarize, we have shown that $\mathcal{C}$ contains the algebra $\mathcal{A}$ and is closed under increasing unions and decreasing intersections. It then follows from the Monotone Class Lemma that $\mathcal{C}$ contains $\langle \mathcal{A} \rangle_\sigma = \mathcal{I} \otimes \mathcal{J}$, so $\mathcal{C} = \mathcal{I} \otimes \mathcal{J}$. $\square$

Def. A collection $\mathcal{C}$ of subsets of some set X is called a monotone class if it is closed under increasing unions and decreasing intersections. The monotone class generated by a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is the $\subseteq$-least monotone class containing $\mathcal{A}$, namely, the intersection of all monotone classes containing $\mathcal{A}$.

Monotone Class Lemma. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a monotone class generated by $\mathcal{A} \subseteq \mathcal{P}(X)$. If $\mathcal{A}$ is an algebra, then $\mathcal{C} = \langle \mathcal{A} \rangle_\sigma$.

Proof. Because any union $C = \bigcup_{n\in\mathbb{N}} C_n$ can be represented as an increasing union $C = \bigcup^{\uparrow}_{m\in\mathbb{N}} \left(\bigcup_{n\le m} C_n \right)$ of finite unions, it is enough to show that $\mathcal{C}$ is an algebra, i.e. is closed under complements and finite unions.

Complements: Let $\mathcal{S} := \{ S \in \mathcal{C} : S^c \in \mathcal{C} \}$. Then $\mathcal{S} \supseteq \mathcal{A}$ because $\mathcal{A}$ is closed under complements and $\mathcal{S}$ is a monotone class because the complement of $\bigcup^{\uparrow}_n$ is $\bigcap^{\downarrow}_n$ and vice versa. Thus, $\mathcal{S} \supseteq \mathcal{C}$ hence $\mathcal{S} = \mathcal{C}$.

Finite unions: We need to show that $\forall U \in \mathcal{C}\ \forall V \in \mathcal{C}$, we have $U \cup V \in \mathcal{C}$. For each $U \in \mathcal{C}$, let

$$\mathcal{S}(U) := \{ V \in \mathcal{C} : U \cup V \in \mathcal{C} \}.$$

Note that $\mathcal{S}(U)$ is a monotone class: if $(V_n) \subseteq \mathcal{S}(U)$, then $U \cup \bigcup^{\uparrow}_{n\in\mathbb{N}} V_n = \bigcup^{\uparrow}_{n\in\mathbb{N}} U \cup V_n \in \mathcal{C}$ and $U \cup \bigcap^{\downarrow}_{n\in\mathbb{N}} V_n = \bigcap^{\downarrow}_{n\in\mathbb{N}} (U \cup V_n) \in \mathcal{C}$ because $\mathcal{C}$ is a monotone class. Also, for all $A \in \mathcal{A}$, $\mathcal{S}(A) \supseteq \mathcal{A}$ because $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A}$ is closed under finite unions. Thus, $\mathcal{S}(A) = \mathcal{C}$ for each $A \in \mathcal{A}$. But then, switching the roles of U and V, we see that for each $V \in \mathcal{C}$, the collection $\mathcal{S}(V)$ contains $\mathcal{A}$ because for each $A \in \mathcal{A}$, $V \in \mathcal{S}(A)$, i.e. $A \cup V \in \mathcal{C}$ hence $V \cup A \in \mathcal{C}$. Thus, $\mathcal{S}(V) = \mathcal{C}$. Hence for all $U, V \in \mathcal{C}$, we have $U \cup V \in \mathcal{C}$. $\square$

We now deduce Fubini-Tonelli for $\mathcal{I} \otimes \mathcal{J}$-measurable functions.

Theorem (Fubini-Tonelli for $\mathcal{I}\otimes\mathcal{J}$). Let $(X,\mathcal{I},\mu)$ and $(Y,\mathcal{J},\nu)$ be σ-finite measure spaces.
Let $f: X\times Y \to \overline{\mathbb{R}} := [-\infty,\infty]$ be a $\mathcal{I}\otimes\mathcal{J}$-measurable function. Then:

(a) $f_x : Y \to \overline{\mathbb{R}}$ and $f^y : X \to \overline{\mathbb{R}}$ are $\mathcal{J}$ and $\mathcal{I}$-measurable for all $x\in X$ and $y \in Y$

(b) Tonelli. If $f \geq 0$, then:

(i) $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are $\mathcal{I}$ and $\mathcal{J}$-measurable.

(ii) $$\int_X \int_Y f_x(y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \int_X f^y(x)\, d\mu(x)\, d\nu(y).$$

(c) Fubini. If f is $\mu\times\nu$-integrable then:

(i) $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are $\mathcal{I}$ and $\mathcal{J}$-measurable and integrable.

(ii) $$\int_X \int_Y f_x(y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \int_X f^y(x)\, d\mu(x)\, d\nu(y).$$